Entanglement and Teleportation of Gaussian States of the Radiation Field

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Received: December 19, 2002

Abstract

We propose a reliable entanglement measure for a two-mode squeezed thermal state of the quantum electromagnetic field in terms of its Bures distance to the set of all separable states of the same kind. The requisite Uhlmann fidelity of a pair of two-mode squeezed thermal states is evaluated as the maximal transition probability between two four-mode purifications. By applying the Peres-Simon criterion of separability we find the closest separable state. This enables us to derive an insightful expression of the amount of entanglement. Then we apply this measure of entanglement to the study of the Braunstein-Kimble protocol of teleportation. We use as input state in teleportation a mixed one-mode Gaussian state. The entangled state shared by the sender (Alice) and the receiver (Bob) is taken to be a two-mode squeezed thermal state. We find that the properties of the teleported state depend on both the input state and the entanglement of the two-mode resource state. As a measure of the quality of the teleportation process, we employ the Uhlmann fidelity between the input and output mixed one-mode Gaussian states.

1 Introduction

Most of the basic achievements in the rapidly developing field of quantum information theory have been obtained for finite-dimensional systems. However, ingenious protocols [1] and successful experiments [2] reported in quantum teleportation of single-mode states of the electromagnetic field justify our present interest in studying entanglement and teleportation of Gaussian field states.

In the present work we review some recent progresses concerning the fidelity of one-mode and two-mode Gaussian states and report our own results. We employ them to get an explicit expression of the *amount of entanglement* of a two-mode squeezed thermal state (STS). As an important application, teleportation of one-mode Gaussian states via a two-mode-STS channel is briefly discussed. We finally point out the properties of the fidelity of teleportation. The paper is organized as follows. Section 2 is devoted to the description of the one-mode and two-mode Gaussian states of the quantum radiation field. Useful formulae for the fidelity of such states are presented in Sec. 3. In Sec. 4, we define the degree of entanglement of a two-mode STS in terms of the Bures distance between the state and the set of all separable STS's: the resulting formula is at the same time simple and insightful. In Sec. 5, we describe by means of characteristic functions (CF's) the teleportation of mixed one-mode states using as resource state an entangled two-mode STS. For Gaussian states, the input-output fidelity is analyzed as an appropriate measure of the efficiency of teleportation .

2 Gaussian states

2.1 One-mode states

Let

$$a = \frac{1}{\sqrt{2}}(q + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip) \tag{2.1}$$

be the amplitude operators of the mode. Any single-mode Gaussian state is a displaced squeezed thermal state (DSTS):

$$\rho = D(\alpha)S(r,\varphi)\rho_T S^\dagger(r,\varphi)D^\dagger(\alpha). \tag{2.2}$$

Here

$$D(\alpha) := \exp\left(\alpha a^\dagger - \alpha^* a\right) \tag{2.3}$$

is a Weyl displacement operator with the coherent-state amplitude $\alpha \in \mathbb{C}$,

$$S(r, \varphi) := \exp\{\frac{1}{2} r[\mathrm{e}^{i\varphi}(a^\dagger)^2 - \mathrm{e}^{-i\varphi} a^2]\} \tag{2.4}$$

is a Stoler squeeze operator with the squeeze factor $r \geq 0$ and squeeze angle $\varphi \in (-\pi, \pi]$, and

$$\rho_T := \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n| \tag{2.5}$$

is a Bose-Einstein density operator with the mean occupancy

$$\bar{n} = \left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]^{-1}. \tag{2.6}$$

The Weyl expansion of the density operator,

$$\rho = \frac{1}{\pi} \int \mathrm{d}^2\lambda \;\; \chi(\lambda) D(-\lambda), \tag{2.7}$$

with $\mathrm{d}^2\lambda := \mathrm{d}\Re e(\lambda)\mathrm{d}\Im m(\lambda)$ points out the one-to-one correspondence between the field state ρ and its CF

$$\chi(\lambda) := \mathrm{Tr}[\rho D(\lambda)]. \tag{2.8}$$

By definition, a Gaussian state has a CF of the form

$$\chi(\lambda) = \exp[-(A + \frac{1}{2})|\lambda|^2 - \frac{1}{2} B^* \lambda^2 - \frac{1}{2} B (\lambda^*)^2 + C^* \lambda - C \lambda^*)], \;\; (A > 0). \tag{2.9}$$

The coefficients A, B, C in the exponent are determined by the DSTS parameters as

$$A = \left(\bar{n} + \frac{1}{2}\right) \cosh(2r) - \frac{1}{2}, \;\; B = -\left(\bar{n} + \frac{1}{2}\right) e^{i\varphi} \sinh(2r), \;\; C = \alpha. \tag{2.10}$$

The covariance matrix,

$$\mathcal{V} := \left(\begin{array}{cc} \sigma(q,q) & \sigma(q,p) \\ \sigma(p,q) & \sigma(p,p) \end{array}\right), \tag{2.11}$$

allows one to write more compactly the generalized Heisenberg uncertainty relation,

$$\det \mathcal{V} \geq \frac{1}{4}, \tag{2.12}$$

as well as the CF (2.9):

$$\chi(\lambda) = \exp\{-\frac{1}{2} X^T \mathcal{V} X - i\, \Xi^T X\}. \tag{2.13}$$

We have denoted:

$$\lambda := -\frac{i}{\sqrt{2}}(x + iy), \quad X^T := (x, y), \tag{2.14}$$

$$\alpha := \frac{1}{\sqrt{2}}(\xi + i\eta), \quad \Xi^T := (\xi, \eta). \tag{2.15}$$

2.2 Nonclassicality

Classical states possess a well-behaved P representation,

$$\rho = \int \mathrm{d}^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|. \tag{2.16}$$

Otherwise, a state is *nonclassical.* For a Gaussian state, the integral

$$P(\alpha) = \frac{1}{\pi^2} \int \mathrm{d}^2\lambda \exp(\alpha\lambda^* - \alpha^*\lambda) \exp(\frac{1}{2}|\lambda|^2)\chi(\lambda) \tag{2.17}$$

exists if and only if $r \leq r_c$, where r_c is the *nonclassicality threshold*,

$$r_c := \frac{1}{2} \ln(2\bar{n} + 1). \tag{2.18}$$

Therefore, a Gaussian state is nonclassical if and only if $r > r_c$.

2.3 Two-mode states

We mention the Weyl expansion of a two-mode state:

$$\rho = \frac{1}{\pi^2} \int \mathrm{d}^2\lambda_1 \mathrm{d}^2\lambda_2 \;\; \chi(\lambda_1, \lambda_2) D_1(-\lambda_1) D_2(-\lambda_2). \tag{2.19}$$

The CF of a Gaussian state has the explicit form

$$\chi(\lambda_1, \lambda_2) = \chi_1(\lambda_1)\chi_2(\lambda_2) \exp\left[-F\lambda_1^*\lambda_2 - F^*\lambda_1\lambda_2^* + G^*\lambda_1\lambda_2 + G\lambda_1^*\lambda_2^*\right]. \quad (2.20)$$

The formula similar to Eq. (2.13)is

$$\chi(\lambda_1, \lambda_2) = \exp\{-\frac{1}{2}X^T\mathcal{V}X - i\,\Xi^T X\} \quad (2.21)$$

with

$$X^T = (x_1, y_1, x_2, y_2), \quad \Xi^T = (\xi_1, \eta_1, \xi_2, \eta_2). \quad (2.22)$$

In Eq. (2.21), $\mathcal{V}$ is the real, symmetric, and positive 4×4 covariance matrix

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{C} \\ \mathcal{C}^T & \mathcal{V}_2 \end{pmatrix}, \quad (2.23)$$

where $\mathcal{V}_j, \quad (j = 1, 2)$ are 2×2 single-mode reduced covariance matrices of the form (2.11), and $\mathcal{C}$ is the cross-covariance matrix,

$$\mathcal{C} = \begin{pmatrix} \sigma(q_1, q_2) & \sigma(q_1, p_2) \\ \sigma(p_1, q_2) & \sigma(p_1, p_2) \end{pmatrix}. \quad (2.24)$$

There are four independent invariants under local symplectic transformations $\mathrm{Sp}(2, \mathbb{R}) \otimes \mathrm{Sp}(2, \mathbb{R})$: $\det\mathcal{V}_1, \quad \det\mathcal{V}_2, \quad \det\mathcal{C}, \quad \det\mathcal{V}$. An inequality that incorporates the Heisenberg uncertainty relations can be expressed in terms of them as

$$\det\mathcal{V} - \frac{1}{4}\left[\det\mathcal{V}_1 + \det\mathcal{V}_2 + 2\det\mathcal{C}\right] + \frac{1}{16} \geq 0. \quad (2.25)$$

An important class of mixed Gaussian states consists of two-mode STS's. Such a state is the unitary transform of a two-mode thermal state,

$$\rho = S_{12}(r, \varphi)(\rho_{T_1} \otimes \rho_{T_2})S_{12}^\dagger(r, \varphi), \quad (2.26)$$

by a two-mode squeeze operator,

$$S_{12}(r, \varphi) := \exp\left[r(\mathrm{e}^{i\varphi}a_1^\dagger a_2^\dagger - \mathrm{e}^{-i\varphi}a_1 a_2)\right]. \quad (2.27)$$

A two-mode STS (2.26) can be experimentally prepared by parametric amplification of light. Its local invariants read:

$$\bar{N}_{1,2} + \frac{1}{2} := \sqrt{\det\mathcal{V}_{1,2}} = \left(\bar{n}_{1,2} + \frac{1}{2}\right)(\cosh r)^2 + \left(\bar{n}_{2,1} + \frac{1}{2}\right)(\sinh r)^2, \quad (2.28)$$

$$\sqrt{-\mathrm{det}\mathcal{C}} = (\bar{n}_1 + \bar{n}_2 + 1)\sinh r \cosh r, \tag{2.29}$$

$$\sqrt{\mathrm{det}\mathcal{V}} = \left(\bar{n}_1 + \frac{1}{2}\right)\left(\bar{n}_2 + \frac{1}{2}\right). \tag{2.30}$$

3 Fidelity

3.1 General properties

Pure states of a quantum mechanical system are described by unit rays in the Hilbert space,

$$f = \{\mathrm{e}^{i\theta}|\Psi\rangle\} \tag{3.1}$$

The squared distance between two unit rays is

$$\begin{aligned}[d(f_1, f_2)]^2 &= \min \||\Psi_1\rangle - \mathrm{e}^{i\theta}|\Psi_2\rangle\|^2 \\ &= 2(1 - |\langle\Psi_1|\Psi_2\rangle|)\end{aligned} \tag{3.2}$$

Consider now a mixed state ρ of a quantum system whose Hilbert space is $\mathcal{H}_A$. A *purification* of ρ is a pure state $|\Phi\rangle\langle\Phi|$ on a tensor product of Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$ such that its reduction to $\mathcal{H}_A$ is the given mixed state:

$$\rho = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|). \tag{3.3}$$

To a pair of mixed states on $\mathcal{H}_A$, $\rho_{1,2}$, one can associate a pair of purifications, $|\Phi_{1,2}\rangle\langle\Phi_{1,2}|$, on $\mathcal{H}_A \otimes \mathcal{H}_B$, which is not unique. The squared Bures distance between ρ_1 and ρ_2, originally introduced on mathematical grounds [3], is

$$d_B^2(\rho_1, \rho_2) := \min \||\Phi_1\rangle - |\Phi_2\rangle\|^2 = 2(1 - \max|\langle\Phi_1|\Phi_2\rangle|). \tag{3.4}$$

Since this definition can obviously be extended to pure states, the set of all quantum states (pure and mixed) may be equipped with the Bures distance to become a metric space. Notice that the "transition probability" between the mixed states ρ_1 and ρ_2, defined later by Uhlmann [4],

$$\mathcal{F}(\rho_1, \rho_2) := \max|\langle\Phi_1|\Phi_2\rangle|^2, \tag{3.5}$$

is closely related to the Bures metric:

$$d_B(\rho_1, \rho_2) = \sqrt{2 - 2\sqrt{\mathcal{F}(\rho_1, \rho_2)}}. \tag{3.6}$$

Uhlmann [4] succeeded in deriving an intrinsic expression of the quantity (3.5), now called *fidelity* [5]:

$$\mathcal{F}(\rho_1, \rho_2) = \left\{ Tr[(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{1/2}] \right\}^2 . \tag{3.7}$$

We list some properties of the fidelity [4, 5, 6, 7, 8, 9]:

1. $0 \leq \mathcal{F}(\rho_1, \rho_2) \leq 1$, and $\mathcal{F}(\rho_1, \rho_2) = 1$ if and only if $\rho_1 = \rho_2$;
2. $\mathcal{F}(\rho_1, \rho_2) = \mathcal{F}(\rho_2, \rho_1)$, (symmetry);
3. $\mathcal{F}(\rho_1, \rho_2) \geq \mathrm{Tr}(\rho_1\rho_2)$; if ρ_1 or/and ρ_2 are pure, then $\mathcal{F}(\rho_1, \rho_2) = \mathrm{Tr}(\rho_1\rho_2)$;
4. $\mathcal{F}(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2) = \mathcal{F}(\rho_1, \rho_2)\mathcal{F}(\sigma_1, \sigma_2)$, (multiplicativity);
5. $\mathcal{F}(U\rho_1 U^\dagger, U\rho_2 U^\dagger) = \mathcal{F}(\rho_1, \rho_2)$, (invariance under unitary transformations);
6. $\sqrt{\mathcal{F}(\rho_1, \rho_2)} = \min_{\{E_b\}} \sum_b \sqrt{\mathrm{Tr}(\rho_1 E_b)}\sqrt{\mathrm{Tr}(\rho_2 E_b)}$, where $\{E_b\}$ is any set of nonnegative operators which is complete, *i. e.* $\sum_b E_b = I$; such a set $\{E_b\}$ is called a positive operator-valued measure (POVM) and $p_{1,2}(b) = \mathrm{Tr}(\rho_{1,2}E_b)$ are probability distributions generated by it.

3.2 One-mode Gaussian states

There are few quantum systems for which an explicit expression for the fidelity of two mixed states is available so far. An important formula has been recently established for one-mode Gaussian states of the radiation field: first Twamley [10] obtained the fidelity of two STS's, and later Scutaru [11] derived the expression for the fidelity of any pair of DSTS's. The latter formula reads:

$$\begin{aligned} F(\rho, \rho') &= \left(\sqrt{\Delta + \Lambda} - \sqrt{\Lambda}\right)^{-1} \\ &\times \exp\left[-\frac{1}{\Delta}\left[(A + A' + 1)|C - C'|^2 - \Re e[(B + B')(C^* - C'^*)^2]\right]\right] \end{aligned} \tag{3.8}$$

with

$$\Delta := \det(\mathcal{V} + \mathcal{V}'), \quad \Lambda := 4\left[\det(\mathcal{V}) - \frac{1}{4}\right]\left[\det(\mathcal{V}') - \frac{1}{4}\right] . \tag{3.9}$$

3.3 Degree of nonclassicality

As an application of the previous formula, Eqs. (3.8) and (3.9), we evaluated the *degree of nonclassicality* of a single-mode Gaussian state ρ [12]. We defined this quantity in terms of the Bures distance between the state ρ and the set $\mathcal{C}_0$ of all classical one-mode Gaussian states:

$$Q_0(\rho) := \frac{1}{2} \min_{\rho' \in \mathcal{C}_0} d_B^2(\rho, \rho'). \tag{3.10}$$

We established the result

$$Q_0(\rho) = 0, \quad (r \leq r_c), \tag{3.11}$$

$$Q_0(\rho) = 1 - [\mathrm{sech}(r - r_c)]^{1/2}, \quad (r > r_c), \tag{3.12}$$

which fulfils the following three requirements:
Q1) $Q_0(\rho)$ vanishes if and only if the state ρ is classical;
Q2) Classical transformations (defined as mapping coherent states into coherent states) preserve $Q_0(\rho)$;
Q3) Nonclassicality does not increase under any POVM mapping.

3.4 The Schmidt decomposition

A pure bipartite state of a composite system is called *separable* or *disentangled* when its state vector is a direct product of two state vectors of the parts:

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle, \tag{3.13}$$

where $|\Psi_A\rangle \in \mathcal{H}_A$ and $|\Psi_B\rangle \in \mathcal{H}_B$. In the opposite case of an *inseparable* or *entangled* pure bipartite state, we point out a very useful biorthogonal expansion of the state vector, which is due to Erhard Schmidt [13]:

$$|\Psi\rangle = \sum_{n=1}^{N} \sqrt{\lambda_n} |\Psi_A^{(n)}\rangle \otimes |\Psi_B^{(n)}\rangle, \tag{3.14}$$

with $\sum_{n=1}^{N} \lambda_n = 1$. The squared Schmidt coefficients λ_n are the common positive eigenvalues of the reductions $\rho_{A,B} := Tr_{B,A}(|\Psi\rangle\langle\Psi|)$. The corresponding eigenvectors, $|\Psi_A^{(n)}\rangle$ in $\mathcal{H}_A$ and $|\Psi_B^{(n)}\rangle$ in $\mathcal{H}_B$, belong to the Schmidt

bases in these spaces. Accordingly, the *Schmidt rank* N cannot exceed the dimensions of the factor Hilbert spaces: $2 \leq N \leq \min(d_1, d_2)$. The value $N = 1$ is excluded because it corresponds to the product state, Eq. (3.13). By definition, $|\Psi\rangle$ is a purification of both ρ_A and ρ_B.

We are interested in the converse problem: Starting from a mixed state ρ_A in $\mathcal{H}_A$, construct purifications in an extended Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_A$ by using the eigenvalue problem of ρ_A and the Schmidt decomposition.

3.5 Two-mode STS's

We have evaluated the fidelity of a pair of two-mode STS's by applying Eq. (3.5) [14]. The eigenvalues of the density operator (2.26) are

$$\lambda_{kl} = \frac{(\bar{n}_1)^k (\bar{n}_2)^l}{(\bar{n}_1 + 1)^{k+1} (\bar{n}_2 + 1)^{l+1}}. \tag{3.15}$$

To a pair of two-mode STS's ρ (parameters: $\bar{n}_1, \bar{n}_2, r, \varphi$) and ρ' (parameters: $\bar{n}_1', \bar{n}_2', r', \varphi'$) we associate the following pair of four-mode purifications written as Schmidt series, Eq. (3.14):

$$|\Psi\rangle = \sum_{kl} \sqrt{\lambda_{kl}} S_{12}(r, \varphi) |kl\rangle \otimes |kl\rangle \tag{3.16}$$

and

$$|\Psi'\rangle = \sum_{mn} \sqrt{\lambda_{mn}'} S_{12}(r', \varphi') |mn\rangle \otimes U |mn\rangle. \tag{3.17}$$

In Eq. (3.17), U is a unitary operator,

$$U := \mathrm{e}^{-i\vartheta} R_{12}(\vartheta) S_{12}(\varrho, \phi), \tag{3.18}$$

whose second factor is a two-mode rotation operator,

$$R_{12}(\vartheta) := \exp\left[-i\vartheta (a_1^\dagger a_1 + a_2^\dagger a_2)\right]. \tag{3.19}$$

U depends on three free variables: the rotation angle ϑ, the squeeze factor ϱ, and the squeeze angle ϕ. We obtained a compact form of the quantum mechanical transition probability between the purifications (3.16) and (3.17).

Its maximum value with respect to the parameters ϑ, ϱ, ϕ yields, according to Eq. (3.5), the fidelity of two arbitrary two-mode STS's:

$$\mathcal{F}(\rho, \rho') = \{[\sqrt{\det(\mathcal{V} + \mathcal{V}')} + (\sqrt{X_1} + \sqrt{X_2})^2]^{1/2} - \sqrt{X_1} - \sqrt{X_2}\}^{-2}, \quad (3.20)$$

where

$$X_{1,2} := \bar{n}_{1,2}\bar{n}'_{1,2}(\bar{n}_{2,1} + 1)(\bar{n}'_{2,1} + 1). \quad (3.21)$$

4 Entanglement

4.1 Inseparable quantum states

In order to include the mixed-state case, Werner [15] gave the following general definition of the separability. A separable (or disentangled) bipartite state ρ_{AB} is a convex combination of product states $\rho_A^{(i)} \otimes \rho_B^{(i)}$:

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \quad p_i \geq 0, \quad \sum_i p_i = 1. \quad (4.1)$$

If the state is not such a mixture, it is termed inseparable (or entangled). For instance, a classical two-mode state of the radiation field is separable, but the converse is not true.

There are two main open problems concerning the entanglement of mixed states. They refer to:

- separability criteria: no universal criterion was still formulated;
- measures of entanglement: no universal entanglement measure could be applied.

4.2 The Peres condition of separability

Peres [16] found a general necessary condition for the separability of a bipartite state: this is the preservation of the nonnegativity of the density matrix under partial transposition. This condition is not a universally sufficient one. However, Horodecki [17] and Simon [18] proved that, in two important cases, Peres' statement is also a criterion for separability. These are, respectively:

a) two-spin-$\frac{1}{2}$ states and also spin-$\frac{1}{2}$-spin-1 states;

b) two-mode Gaussian states of the radiation field.

In the latter case, Simon [18] gave a $\mathrm{Sp}(2,\mathbb{R}) \otimes \mathrm{Sp}(2,\mathbb{R})$ invariant form of the separability criterion:

$$\det\mathcal{V} - \frac{1}{4}\left[\det\mathcal{V}_1 + \det\mathcal{V}_2 + 2|\det\mathcal{C}|\right] + \frac{1}{16} \geq 0. \tag{4.2}$$

Using Eq. (4.2), one can easily check whether a two-mode Gaussian state is separable or not [19]. In particular, for two-mode STS's, the Peres-Simon criterion (4.2) reads:

$$(\cosh r)^2 \leq (\cosh r_s)^2 := \frac{(\bar{n}_1+1)(\bar{n}_2+1)}{\bar{n}_1+\bar{n}_2+1}. \tag{4.3}$$

4.3 Measures of entanglement

In their search for a good measure of entanglement, $E(\rho)$, of a bipartite state ρ, Vedral *et al* stated in Ref. [20] the following general demands:
E1) $E(\rho)$ vanishes if and only if the state ρ is separable;
E2) local unitary transformations preserve $E(\rho)$;
E3) $E(\rho)$ does not increase under local general measurements.
They proved that a convenient entanglement measure could be a distance between the state ρ and the set $\mathcal{D}$ of all the separable states of the given system:

$$E(\rho) := \min_{\sigma\in\mathcal{D}} d(\rho,\sigma). \tag{4.4}$$

Two candidates for the distance $d(\rho,\sigma)$ in Eq. (4.4) are found to be suitable:

- The quantum relative entropy,

  $$S(\sigma||\rho) := Tr[\sigma(\ln\sigma - \ln\rho)], \tag{4.5}$$

  which is not a true metric;

- The Bures metric, Eq. (3.4).

Vedral *et al* [20] succeeded in obtaining an explicit expression of the amount of entanglement $E(\rho)$ of any *pure* bipartite state ρ, by choosing as "distance" in Eq. (4.4) the relative entropy, Eq. (4.5). For $d(\rho,\sigma) = S(\sigma||\rho)$ they found that the measure of its entanglement is the common von Neumann entropy of the reduced states ρ_A and ρ_B:

$$E(\rho) = S(\rho_A) := -Tr_A[\rho_A \ln(\rho_A)]. \tag{4.6}$$

4.4 Entanglement of a two-mode STS

In order to estimate the amount of entanglement of a two-mode STS, we apply Eq. (4.4) using the Bures distance. However, we make an upper bound approximation by replacing the set $\mathcal{D}$ of all the separable two-mode states by its subset $\mathcal{D}_0$ consisting of all the separable STS's. Hence for a given inseparable two-mode STS, ρ, we evaluate the following degree of entanglement:

$$E_0(\rho) := \min_{\rho' \in \mathcal{D}_0} \frac{1}{2} d_B^2(\rho, \rho') = 1 - \max_{\rho' \in \mathcal{D}_0} \sqrt{\mathcal{F}(\rho, \rho')}. \tag{4.7}$$

We make use of the fidelity (3.20) of the given inseparable STS ρ (parameters : $\bar{n}_1, \bar{n}_2, r > r_s, \varphi$) with respect to an arbitrary separable STS $\rho' \in \mathcal{D}_0$

(parameters : $\bar{n}_1', \bar{n}_2', r', \varphi'$, with $r' \leq r_s{}'$). We determine the parameters of the closest separable STS $\tilde{\rho}$: $\tilde{\varphi} = \varphi$, $\tilde{r} = \tilde{r}_s$, $\tilde{\bar{n}}_1$, $\tilde{\bar{n}}_2$, and then get a significant formula [21],

$$E_0(\rho) = 0, \quad (r \leq r_s), \tag{4.8}$$

$$E_0(\rho) = 1 - \operatorname{sech}(r - r_s), \quad (r > r_s), \tag{4.9}$$

which observes the demands E1)-E3) for an adequate measure of entanglement.

5 Teleportation

5.1 Spin-$\frac{1}{2}$ states

The discovery of the possibility of quantum teleportation by Bennett *et al* [22] opened new research directions in the field of quantum processing of information. We quote from Ref.[2]: "Quantum teleportation is the disembodied transport of an unknown quantum state from one place to another". The key idea of the *Gedankenexperiment* described in Ref.[22] is that two distant operators, Alice at a sending station and Bob at a receiving terminal, share an entangled quantum bipartite state and exploit its *nonlocal* character as a quantum resource. The resource state, which is also called an Einstein-Podolsky-Rosen (EPR) state [23], is here the singlet state of a pair of spin-$\frac{1}{2}$ particles. Particle 1 is given to Alice and particle 2 is given

to Bob. Alice intends to transport an *unknown* state of a third spin-$\frac{1}{2}$ particle to Bob. She performs a complete projective measurement on the joint system and then conveys its outcome to Bob via a classical communication channel. As a consequence of Alice's measurement, the total-spin state of the three-particle system collapses. Due to the entanglement, this involves a breakdown of the spin-$\frac{1}{2}$ state of Bob's particle 2. Nevertheless, Bob makes use of the information transmitted classically by Alice to transform his reduced state into an output that is an accurate replica of the original unknown input.

5.2 One-mode states of the radiation field

Along the lines sketched above, Braunstein and Kimble [1] put forward a teleportation protocol for optical one-mode field states. They propose as resource state shared by Alice and Bob a two-mode squeezed vacuum state (SVS). Very soon, this protocol was implemented into a successful experiment that demonstrated the quantum teleportation of optical coherent states [2]. It is useful to present briefly the Braunstein-Kimble protocol. It consists of three steps, as follows.

• Step 1. Alice mixes two waves with a beam splitter, namely an unknown one-mode input in the state ρ_{in} and the two-mode beam in the EPR state ρ_{AB}.

• Step 2. Alice measures simultaneously the observables $q = q_{in} - q_A, \;\; p = p_{in} + p_A$ in the resulting three-mode state. Quantum Mechanics predicts their distribution function:

$$\mathcal{P}(q,p) = \mathrm{Tr}_{in,AB}\left[P(\rho_{in} \otimes \rho_{AB})P^{\dagger}\right] \tag{5.1}$$

with

$$P = |\Phi_{in,A}(q,p)\rangle\langle\Phi_{in,A}(q,p)| \otimes I_B. \tag{5.2}$$

The complete von Neumann measurement performed by Alice entails a collapse of the tripartite state to a state whose reduction at Bob's disposal is

$$\rho'_B = \frac{1}{\mathcal{P}(q,p)}\mathrm{Tr}_{in,A}\left[P(\rho_{in} \otimes \rho_{AB})P^{\dagger}\right]. \tag{5.3}$$

• Step 3. Using classical communication lines, Alice conveys to Bob the outcome $\{q,p\}$ of her measurement. Then, Bob superposes a coherent field

whose amplitude is precisely $\mu = \frac{1}{\sqrt{2}}(q + ip)$ on the mode ρ'_B at his hand:

$$\rho'_B \longrightarrow \rho''_B = D_2(\mu)\rho'_B D_2^\dagger(\mu). \tag{5.4}$$

A more realistic ensemble description of the projective measurements carried out by Alice yields the teleported state

$$\rho_{out} = \int\int \mathrm{d}q\mathrm{d}p\mathcal{P}(q,p)\rho''_B. \tag{5.5}$$

5.3 CF of the teleported state

The common eigenfunction of the pair of continuous quantum variables $\{q, p\}$ measured by Alice,

$$|\Phi_{in,A}(q,p)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\eta \mathrm{e}^{ip\eta}|q+\eta\rangle_{in} \otimes |\eta\rangle_A, \tag{5.6}$$

has the coherent-state expansion

$$\begin{aligned}|\Phi_{in,A}(q,p)\rangle &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{|\mu|^2}{2} - \frac{ipq}{2}\right] \frac{1}{\pi^2} \int\int \mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, |\alpha\rangle_{in} \otimes |\beta\rangle_A \\ &\times \exp\left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta^* - \mu^*\beta^* + \mu\alpha^*\right].\end{aligned} \tag{5.7}$$

Using the Weyl expansions of the density operators $\rho_{in}, \rho_{AB}, \rho'_B, \rho_{out}$, we get the CF of the state at Bob's hand after the phase-space translation performed by him:

$$\chi_{out}(\lambda) = \chi_{in}(\lambda)\chi_{AB}(\lambda^*, \lambda). \tag{5.8}$$

Equation (5.8) is the main result of the present work. Therefore, if the EPR state ρ_{AB} is a two-mode Gaussian state, then any single-mode Gaussian input is teleported as a single-mode Gaussian output.

In what follows we choose as EPR state an entangled two-mode *symmetric* STS. This is a state (2.26) having equal mode frequencies and possessing equal mean numbers of thermal photons in both modes, $\bar{n}_1 = \bar{n}_2 < \frac{1}{2}(e^{2r} - 1)$; in addition, we assume that the squeeze angle is equal to zero:

$$\rho_{AB} = S_{12}(r, 0)(\rho_T \otimes \rho_T)S_{12}^\dagger(r, 0). \tag{5.9}$$

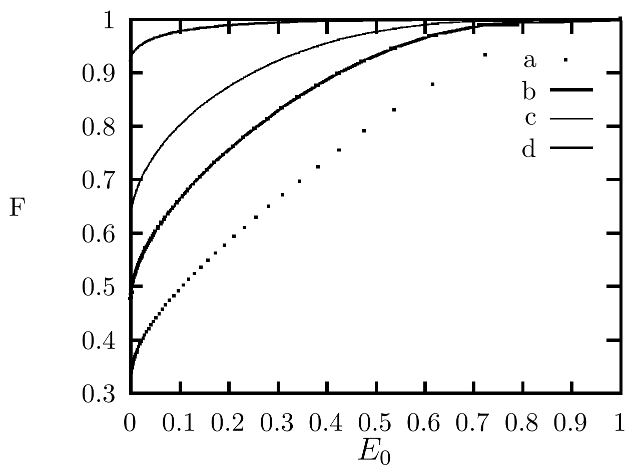

Figure 1: Fidelity of teleportation for several mixed Gaussian states versus the entanglement of the EPR state (5.9). The squeeze factor is $r = 1$. The curves a ($\bar{n} = 0$), b ($\bar{n} = 0.1$), c ($\bar{n} = 0.5$) correspond to nonclassical input states while the d one is for the classical state with ($\bar{n} = 5$). It is clear that the fidelity increases with the degree of mixing.

Then Eq. (5.8) shows that teleportation of a Gaussian state merely provides additional noise, exactly like superposition of a thermal field:

$$A_{out} = A_{in} + \exp\left[-2(r - r_s)\right], \quad B_{out} = B_{in}, \quad C_{out} = C_{in}. \tag{5.10}$$

5.4 Fidelity of teleportation

The quality of the teleportation protocol is quantified by the output-input fidelity, called fidelity of teleportation. In particular, according to Eqs. (3.8) and (3.9), the fidelity of teleportation of pure or mixed one-mode Gaussian states via the EPR state (5.9) is

$$\mathcal{F}(\rho_{out}, \rho_{in}) = \left(\sqrt{\Delta + \Lambda} - \sqrt{\Lambda}\right)^{-1}, \tag{5.11}$$

where

$$\Delta = 4\left(y^2 + xyz + \frac{1}{4}z^2\right), \quad \Lambda = 4\left(y^2 - \frac{1}{4}\right)\left(y^2 - \frac{1}{4} + 2xyz + z^2\right), \tag{5.12}$$

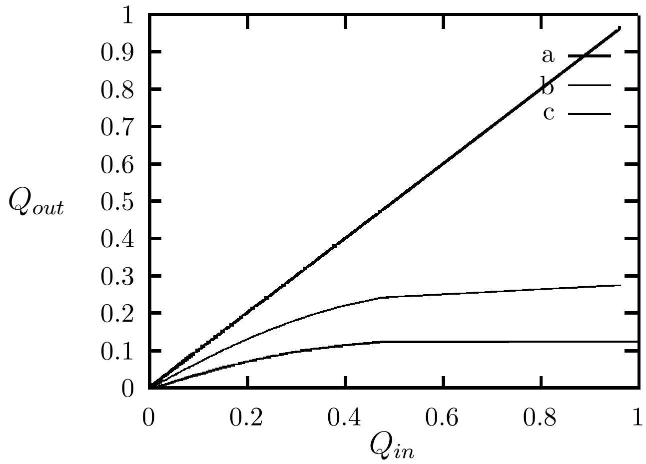

Figure 2: Degree of nonclassicality of the teleported Gaussian state versus the similar quantity of the input state plotted for several values of the entanglement of the EPR state (5.9) : $E_0 = 1$ (curve a),$E_0 = 0.615$ (curve b), $E_0 = 0.425$ (curve c). Both measures of nonclassicality and entanglement are defined using the Bures distance, Eqs. (3.12) and (4.9), respectively.

with the variables

$$x := \cosh(2r_{in}) \geq 1, \quad y := \bar{n}_{in} + \frac{1}{2} \geq \frac{1}{2}, \quad z := \exp\left[-2(r - r_s)\right] > 0. \quad (5.13)$$

Note that the parameters x and y are characteristics of the input one-mode DSTS, while z depends only on the degree of entanglement, Eq. (4.9), of the EPR state (5.9).

We mention some properties of the fidelity of teleportation

$$\mathcal{F}(\rho_{out}, \rho_{in}) := \mathcal{F}(x, y, z). \quad (5.14)$$

- $\frac{\partial \mathcal{F}}{\partial x} < 0$: Fidelity decreases with the squeeze factor of the input state.
- $\frac{\partial \mathcal{F}}{\partial y} > 0$: Fidelity decreases with the degree of purity of the input state.

Accordingly, for $r_{in} > r_c$, fidelity decreases with the degree of nonclassicality.

- $\frac{\partial \mathcal{F}}{\partial z} < 0$: Fidelity increases with the entanglement of the STS resource.

Indeed,

$$E_0(\rho_{AB}) = \frac{(1-\sqrt{z})^2}{1+z}, \quad (z < 1 \Longleftrightarrow r > r_s). \tag{5.15}$$

In the special case of an input coherent state, we recover the formula [24]

$$\mathcal{F}(1, \frac{1}{2}, z) = \frac{1}{1+z}, \tag{5.16}$$

so that

$$\mathcal{F}(1, \frac{1}{2}, z) > \frac{N}{N+1} \Longleftrightarrow r > r_s + \frac{1}{2}\ln N, \quad (N = 1, 2, 3, ...). \tag{5.17}$$

Acknowledgment

This work was supported by the Romanian CNCSIS through a grant for the University of Bucharest.

References

[1] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[2] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).

[3] D. Bures, Trans. Am. Math. Soc. **135**, 199 (1969).

[4] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976); Rep. Math. Phys. **24**, 229 (1986).

[5] R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).

[6] S. L. Braunstein and C.M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).

[7] C. A. Fuchs, PhD thesis, University of New Mexico, Albuquerque, NM, 1995: quant-ph/9601020/1996.

[8] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996); H. Barnum, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. A **54**, 4707 (1996).

[9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK, 2000, pp. 409-415.

[10] J. Twamley, J. Phys. A: Math. Gen. **29**, 3723 (1996).

[11] H. Scutaru, J. Phys. A: Math. Gen. **31**, 3659 (1998); Gh.-S. Părăoanu and H. Scutaru, Phys. Rev. A **58**, 869 (1998).

[12] Paulina Marian, T. A. Marian, and H. Scutaru, Phys. Rev. Lett. **88**, 153601 (2002).

[13] E. Schmidt, Math. Annalen **63**, 433 (1906).

[14] Paulina Marian and T. A. Marian, in preparation.

[15] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[16] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[17] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[18] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[19] Paulina Marian, T. A. Marian, and H. Scutaru, J. Phys. A: Math. Gen. **34**, 6969 (2001).

[20] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997); V. Vedral, M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).

[21] Paulina Marian, T. A. Marian, and H. Scutaru, Phys. Rev. A **68**, 062309 (2003).

[22] C. H. Bennett, G. Brassard, C. Crépeau, R.Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[23] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[24] T. Ide, H. F. Hofmann, A. Furusawa, and T. Kobayashi, Phys. Rev. A **65**, 062303 (2002)